Topological spaces.

Def. A topology on a set X is a collection $\mathcal{T}$ of subsets of X satisfying:

(i) $\{\emptyset, X\} \subseteq \mathcal{T}$;

(ii) $\mathcal{T}$ is closed under arbitrary unions and finite intersections.

The pair $(X, \mathcal{T})$ is called a topological space and the sets in $\mathcal{T}$ are called open. The complements of open sets are called closed.

Example. For a metric space (X, d), the collection of d-open sets is a topology on X, which we call the topology induced by d.

Def. A topological space $(X, \mathcal{T})$ is called metrizable if $\mathcal{T}$ is induced by some metric on X.

Remark. Many important topological spaces are not metrizable, and this is why this new subject is important. E.g. the topology of pointwise convergence is not metrizable in general, but it's a really important notion of convergence.

Remark. All notions and proofs done for metric spaces which only used open/closed sets remain valid for arbitrary topological spaces.

Def. A topological space X is called Hausdorff (the mathematician who contributed most in the foundations of this subject) if any pair $x, y \in X$ of distinct points admit disjoint open neighbourhoods $U \ni x$ and $V \ni y$.
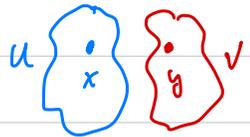
Example. All metric spaces are Hausdorff because $\forall\, x \neq y$, $d(x, y) > 0$ so open balls centered

at x and y, resp., of radii $\frac{1}{2} d(x,y)$ are disjoint.

Examples of non-Hausdorff top. spaces (hence non-metrizable).

(a) For any set X, the **trivial** topology on X is $T_o := \{\emptyset, X\}$. If $|X| \geq 2$, T_o is non-Hausdorff. However, the "opposite" topology, namely, the **discrete topology**, on X $T_{disc} := \mathcal{P}(X)$, is metrizable by the $0/1$-metric.

(b) Let $X := \{0,1\}$ with the **half-open topology** $T_h := \{\emptyset, X, \{0\}\}$. This is non-Hausdorff.
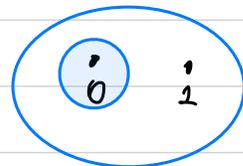
(c) Let X be a set and let $T_{cofin} := \{U \subseteq X : \overset{U \text{ is cofinite}}{\overline{X \setminus U \text{ is finite}}}\} \vee \{\emptyset\}$

Since finite unions of finite sets is finite, T_{cofin} is closed under finite intersections. It is also trivially closed under arbitrary unions, and contains $\{X, \emptyset\}$, hence T_{cofin} is a topology on X called the **cofinite topology**.

When X is finite, $T_{cofin} = T_{disc}$, the discrete topology.

When X is infinite (e.g. $X = \mathbb{N}$ or $X = \mathbb{R}$), T_{cofin} is not Hausdorff because the intersection of any two cofinite sets is cofinite, hence nonempty.

(d) Let $\mathbb{F}$ be a field, e.g. $\mathbb{Q}$; $\mathbb{R}$; $\mathbb{Z}/p\mathbb{Z}$, p prime; $\mathbb{C}$. Let $X := \mathbb{F}^n$ for $n \geq 1$.
We define the **Zariski topology** T_Z on X by declaring the following sets closed: the sets of solutions $\vec{x} \in \mathbb{F}^n$ to finite systems of polynomial equations in n variables, i.e. the closed sets are exactly of the form:
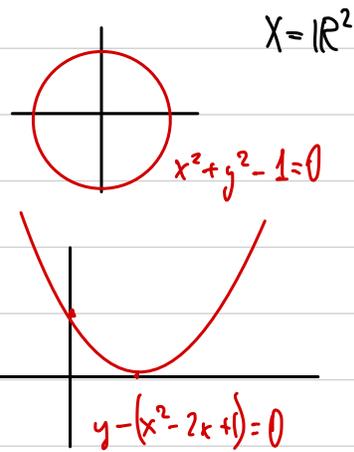$$\{\vec{x} \in \mathbb{F}^n : p_1(\vec{x}) = 0, \ p_2(\vec{x}) = 0, \ldots, p_k(\vec{x}) = 0\},$$
where the p_i are polynomials in n variables.

That the collection of closed sets $\hat{T}_Z$ is closed under finite unions follows from the fact that for polynomials $P_1, P_2 \in \mathbb{F}[t_1,\ldots,t_n]$, the union of the sets of roots of p_1 and p_2

$X = \mathbb{R}^2$

$x^2 + y^2 - 1 = 0$

$y - (x^2 - 2x + 1) = 0$

$\{(0,1),(1,0)\}$ is closed.

is exactly the set of roots of $P_1 \cdot P_2$. However, the fact that $\hat{\mathcal{T}}_Z$ is closed under arbitrary intersections is highly non-trivial, and this is called the **Hilbert basis theorem** (which says in the algebraic language that in the ring $\mathbb{F}[t_1,..,t_n]$, all ideals are finitely generated).

    This topology is non-Hausdorff unless the field $\mathbb{F}$ is finite; this is because every closed set has less dimension than the space, so all nonempty open sets have nonempty intersection (in fact the complement of this intersection has dimension $< n$).

    <u>Note:</u> For $n=1$, the Zariski closed sets are exactly all finite sets because every polynomial of one variable of degree k has $\leq k$ roots and for each $a \in \mathbb{F}$, the set of roots of the polynomial $p(t) := t-a$ is $\{a\}$. In other words, the Zariski top on $\mathbb{F}$ is exactly the cofinite topology $\mathcal{T}_{cofin}$.

<u>**Obs.**</u> An arbitrary intersection of topologies on a set X is still a topology on X.

<u>**Def.**</u> Let X be a set and $\mathcal{E} \subseteq \mathcal{P}(X)$. The **topology generated by** $\mathcal{E}$ is the smallest topology on X containing $\mathcal{E}$, namely, the intersection of all topologies containing $\mathcal{E}$ (since there is at least one, namely, the discrete top. $\mathcal{P}(X)$). Denote this by $\mathcal{T}_{\mathcal{E}}$.

This is an non-constructive top-down definition, but luckily, there is a constructive bottom-up equivalent:

<u>**Prop.**</u> For any set X and $\mathcal{E} \subseteq \mathcal{P}(X)$, the topology $\mathcal{T}_{\mathcal{E}}$ generated by $\mathcal{E}$ consists of all (arbitrary) unions of finite intersections of sets in $\mathcal{E}$, i.e.
$$\mathcal{T}_{\mathcal{E}} = \{\emptyset, X\} \cup \{\text{arbitrary unions of sets in } \mathcal{E}_{\cap}\} =: \mathcal{E}_{\cup\cap}$$
where $\mathcal{E}_\cap := \{\text{finite intersections of sets in } \mathcal{E}\}$.

<u>Proof.</u> First observe that since $\mathcal{T}_{\mathcal{E}} \supseteq \mathcal{E}$, we also have $\mathcal{T}_{\mathcal{E}} \supseteq \mathcal{E}_\cap$ because finite intersections of open sets are open. But then also $\mathcal{T}_{\mathcal{E}} \supseteq \mathcal{E}_{\cup\cap}$ because arbitrary unions of open sets are open. To show $\mathcal{T}_{\mathcal{E}} \subseteq \mathcal{E}_{\cup\cap}$, we just need to show that $\mathcal{E}_{\cup\cap}$ is

a topology, by the def of T_2. But clearly, $\Sigma_{U \textcircled{A}}$ is closed under arbitrary unions, by definition, and it is also closed under finite intersections by the distributivity of intersections through unions:

$$\left(\underbrace{\bigcup_{i \in I} \bigcap_{j < k_i} E_{ij}}_{\in \Sigma_{\textcircled{A}}} \right) \wedge \left(\underbrace{\bigcup_{n \in N} \bigcap_{\ell < M_n} F_{n\ell}}_{\in \Sigma_{\textcircled{A}}} \right) = \underbrace{\bigcup_{\substack{i \in J \\ n \in N}} \bigcap_{j < k_i} E_{ij} \wedge \bigcap_{\ell < m_n} F_{n\ell}}_{\in \Sigma_{\textcircled{A}}}, \text{ where}$$

the sets E_{ij} and $F_{n\ell} \in \Sigma$. $\boxed{\text{QED}}$